STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 1

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 1.1. The process $X_t = \sigma W_t + bt$ with $\sigma, b \in \mathbb{R}$ and W Brownian motion is observed at the time points $0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = T_n$. Denote $\Delta X_{i,n} = X_{t_{i,n}} - X_{t_{i-1,n}}$ and $\Delta t_{i,n} = t_{i,n} - t_{i-1,n}$.

a) Compute the MLE $\hat{\theta}_{\mathrm{MLE}}$ for the parameter $\theta = (b, \sigma^2)$ and find conditions such that the MLE is consistent, i.e., $\hat{\theta}_{\mathrm{MLE}} \xrightarrow{d} \theta$.

b) Assume that b is known. Compute the Fisher information for the parameter σ^2.

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 2

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 2.1. Consider the SDE $\mathrm{d}X_t = aX_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$, $t \geq 0$, $X_0 = X^{(0)} \in L^2$. Make the Ansatz $X_t(\omega) = C_t(\omega)e^{at}$. Apply the Itô formula to $C_t(\omega)$ and derive the solution of the SDE in this way. For $a < 0$ show that $X_t \xrightarrow{d} N(0, -\sigma^2/(2a))$ as $t \to \infty$. For $a < 0$ find $X^{(0)}$ so that the solution of the SDE is stationary.

Ex. 2.2. a) Use the Itô formula to show $\int_0^t W_s\,\mathrm{d}W_s = \frac{1}{2}(W_t^2 - t)$ for Brownian motion W.

b) Let $\hat{a}_T$ be the MLE in the Ornstein–Uhlenbeck model with time-continuous observations $(X_t)_{t\in[0,T]}$ and initial condition $X^{(0)} = 0$. Consider $\hat{a}_T$ as $T \to \infty$ under $\mathbb{P}^0$, i.e., with true parameter $a = 0$. Show that $\hat{a}_T$ is consistent and that $T\hat{a}_T$ converges in distribution. Show that the limit is not a centred normal distribution.

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 3

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 3.1. Consider the regression model $Y_i = f(i/n) + \epsilon_i$, $i = 1, \ldots, n$, where ϵ_i are i.i.d. errors with $\mathbb{E}[\epsilon_i] = 0$ and $\mathrm{Var}(\epsilon_i) < \infty$. Let $f : [0,1] \to \mathbb{R}$ be differentiable and $\|f'\|_\infty \leq M$. For $x \in [0,1]$ define the estimator $\hat{f}_n(x,h)$ by

$$\hat{f}_n(x,h) = \frac{\sum_{i=1}^n Y_i \mathbb{1}_{[x-h,x+h]}(i/n)}{\sum_{i=1}^n \mathbb{1}_{[x-h,x+h]}(i/n)} \qquad \text{for } \sum_{i=1}^n \mathbb{1}_{[x-h,x+h]}(i/n) \neq 0$$

and $\hat{f}_n(x,h) = 0$ otherwise. Show that $|\hat{f}_n(x, n^{-1/3}) - f(x)| = O_{\mathbb{P}}(n^{-1/3})$.

Ex. 3.2. Let $\mathrm{d}X_t = b(t)\,\mathrm{d}t + \frac{\sigma}{\sqrt{n}}\,\mathrm{d}W_t$, $t \in [0,1]$, $X_0 = 0$, where $\sigma > 0$, $b : [0,1] \to \mathbb{R}$ and W is Brownian motion. For time-continuous observations $(X_t)_{t\in[0,1]}$ we define the estimator

$$\hat{b}_n(x,h) = \frac{\int_0^1 \mathbb{1}_{[x-h,x+h]}(t)\,\mathrm{d}X_t}{\int_0^1 \mathbb{1}_{[x-h,x+h]}(t)\,\mathrm{d}t}.$$

Show that for α-Hölder continuous functions b with $\alpha \in (0,1]$, for $h = n^{-1/(2\alpha+1)}$ and $x \in [0,1]$

$$|\hat{b}_n(x,h) - b(x)| = O_{\mathbb{P}}(n^{-\alpha/(2\alpha+1)}).$$

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 4

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 4.1. Let b, σ and $1/\sigma$ be continuous and bounded. Let $s(x) = \int_0^x \exp(-\int_0^y 2b(z)/\sigma(z)^2 \,\mathrm{d}z)\,\mathrm{d}y$ and assume $s(x) \to \pm\infty$ as $x \to \pm\infty$. Let $(X_t)_{t\geq 0}$ satisfy the SDE $\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t$ with $X_0 = X^{(0)}$.

The adjoint of the infinitesimal generator A is given by $A^*g = -(bg)' + \frac{1}{2}(\sigma^2 g)''$ for $g \in C_0^2(\mathbb{R})$. From the theory of semigroups it follows that if we can find any non-negative $m \in C_0^2(\mathbb{R})$, $m \not\equiv 0$, which has finite integral and satisfies $A^*m = 0$, then m is up to normalising the density of a stationary distribution.

a) Show that $Y_t = s(X_t)$ satisfies $\mathrm{d}Y_t = \tilde{\sigma}(Y_t)\,\mathrm{d}W_t$ with $\tilde{\sigma}(y) = s'(s^{-1}(y))\sigma(s^{-1}(y))$.

b) Let $\tilde{A}^*$ be adjoint of the infinitesimal generator of the SDE in part a). Find a non-negative function $m \not\equiv 0$ satisfying $\tilde{A}^*m = 0$.

c) Assume $\tilde{G} = \int_{-\infty}^{\infty} 1/\tilde{\sigma}(y)^2\,\mathrm{d}y < \infty$ with $\tilde{\sigma}$ is as in part a). Let Y be a random variable having density $1/(\tilde{G}\tilde{\sigma}(y)^2)$. Determine the density of $X = s^{-1}(Y)$ in terms of σ and b.

Ex. 4.2. Consider the linear factor model $\mathrm{d}X_t = \vartheta b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t$ with ϑ unknown. Let b and σ be known, measurable and bounded and let $\inf_{x\in\mathbb{R}} \sigma^2(x) \geq \underline{\sigma}^2 > 0$. Let there be $M, \gamma > 0$ such that $\operatorname{sign}(x)\frac{2b}{\sigma^2}(x) \leq -\gamma$ for all x with $|x| \geq M$. Let $(X_t)_{t\in[0,T]}$ be time-continuous observations of a stationary solution X of the SDE with $\vartheta > 0$. Let $\mathbb{E}[b(X_0)^2/\sigma(X_0)^2] > 0$. Show that

$$\left(\int_0^T \frac{b(X_t)}{\sigma(X_t)}\,\mathrm{d}W_t\right) \Big/ \left(\int_0^T \frac{b(X_t)^2}{\sigma(X_t)^2}\,\mathrm{d}t\right)$$

has positive denominator with probability tending to one and is of order $O_{\mathbb{P}}(1/\sqrt{T})$.

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 5

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 5.1. Let b and σ be measurable and locally bounded and let $\inf_{x\in\mathbb{R}} \sigma^2(x) \geq \underline{\sigma}^2 > 0$. Let there be $M, \gamma > 0$ such that $\operatorname{sign}(x) 2b(x)/\sigma(x)^2 \leq -\gamma$ for all x with $|x| \geq M$. Let X be a stationary solution and μ be an invariant measure of the SDE $\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t$. Define $Af(x) := \frac{1}{2}\sigma^2(x) f''(x) + b(x) f'(x)$. For $f, g \in C^2(\mathbb{R})$ with compact support show that A is symmetric with respect to μ, i.e.,

$$\int_{-\infty}^{\infty} Af(x) g(x)\,\mathrm{d}\mu(x) = \int_{-\infty}^{\infty} f(x) Ag(x)\,\mathrm{d}\mu(x).$$

Ex. 5.2. Consider the process $X_t = x_0 + \int_0^t \sigma(s)\,\mathrm{d}W_s$ for $t \in [0, T]$, where $T > 0$ fixed, $x_0 \in \mathbb{R}$, W is Brownian motion and $\sigma : [0, T] \to \mathbb{R}$ is a bounded deterministic function. For $n \geq 1$ the process $(X_t)_{t\in[0,T]}$ is observed at the times $0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = T$. Denote $\Delta X_{i,n} = X_{t_{i,n}} - X_{t_{i-1,n}}$, $\Delta t_{i,n} = t_{i,n} - t_{i-1,n}$ and $\Delta_n = \max_{1\leq i\leq n} \Delta t_{i,n}$. Let $g : [0, T] \to \mathbb{R}$, $\alpha \in (0, 1]$ and $R > 0$ be such that $|g(t) - g(s)| \leq R|t - s|^\alpha$ and $|g(t)| \leq R$ for all $s, t \in [0, T]$. Consider the estimator $\hat{\Lambda}_n(g) = \sum_{i=1}^n g(t_{i-1,n})(\Delta X_{i,n})^2$ for $\Lambda(g) = \int_0^T g(s)\sigma(s)^2\,\mathrm{d}s$.

a) Define $M_n := \sum_{i=1}^n g(t_{i-1,n}) \left((\Delta X_{i,n})^2 - \int_{t_{i-1,n}}^{t_{i,n}} \sigma(s)^2\,\mathrm{d}s \right)$. Show that there exists a constant $C > 0$ depending only on R and T such that $\mathbb{E}[M_n^2] \leq C\|\sigma^4\|_\infty \Delta_n$.

b) Show that there exists a constant $D > 0$ depending only on R and T such that

$$\mathbb{E}[(\hat{\Lambda}_n(g) - \Lambda(g))^2] \leq D\|\sigma^4\|_\infty \max(\Delta_n, \Delta_n^{2\alpha}).$$

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 6

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 6.1. Consider the setting of Ex. 1.1 with $n \to \infty$, $T_n \to \infty$ and both $b \in \mathbb{R}$ and $\sigma^2 > 0$ unknown. Construct confidence intervals B_n and S_n for b and σ^2, respectively, which are asymptotic of level $\alpha \in (0,1)$, i.e.,

$$\mathbb{P}(b \in B_n) \to 1 - \alpha \quad \text{as } n \to \infty,$$
$$\mathbb{P}(\sigma^2 \in S_n) \to 1 - \alpha \quad \text{as } n \to \infty.$$

Ex. 6.2. In the setting of Ex. 3.1 let $x \in (0,1)$, $nh_n \to \infty$ and $nh_n^3 \to 0$ as $n \to \infty$. Define $\sigma^2 := \mathrm{Var}(\epsilon_1)$.

a) Show that $\sqrt{nh_n}(\hat{f}_n(x, h_n) - f(x)) \to N(0, \sigma^2/2)$ as $n \to \infty$.

b) With σ^2 known construct confidence intervals I_n for $f(x)$ which are of asymptotic level $\alpha \in (0,1)$, i.e.,
$$\mathbb{P}(f(x) \in I_n) \to 1 - \alpha \quad \text{as } n \to \infty.$$

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 7

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 7.1. Let X be a d-dimensional Lévy process with Lévy measure ν.

a) Let $\int_{\mathbb{R}^d} |x| \mathbb{1}_{\{|x|>1\}} \, \mathrm{d}\nu(x) < \infty$. Show that $\mathbb{E}[X_t] = \gamma_1 t$ with $\gamma_1 = \gamma + \int_{\mathbb{R}^d} x \mathbb{1}_{\{|x|>1\}} \, \mathrm{d}\nu(x)$.

b) Let $d = 1$ and $\int_{\mathbb{R}} x^2 \, \mathrm{d}\nu(x) < \infty$. Show that $\mathbb{E}[X_t] = \gamma_1 t$ for γ_1 as in a) and $\mathrm{Var}(X_t) = (\sigma^2 + \tilde{\nu}(\mathbb{R}))t$ for $\mathrm{d}\tilde{\nu}(x) = x^2 \, \mathrm{d}\nu(x)$.

Ex. 7.2. Let φ be the characteristic function of a random variable X on $\mathbb{R}$ and φ_n the empirical characteristic function of n i.i.d. samples of X. For complex-valued random variables Z_i we define $\mathrm{Cov}_{\mathbb{C}}(Z_1, Z_2) = \mathbb{E}[Z_1 \bar{Z}_2] - \mathbb{E}[Z_1]\overline{\mathbb{E}[Z_2]}$ and $\mathrm{Var}_{\mathbb{C}}(Z_1) = \mathbb{E}[|Z_1 - \mathbb{E}[Z_1]|^2]$.

a) Show that $\mathbb{E}[\varphi_n(u)] = \varphi(u)$, $\mathrm{Cov}_{\mathbb{C}}(\varphi_n(u), \varphi_n(v)) = \frac{1}{n}(\varphi(u-v) - \varphi(u)\varphi(-v))$, $\mathrm{Var}_{\mathbb{C}}(\varphi_n(u)) = \frac{1}{n}(1 - |\varphi(u)|^2) \leq \frac{1}{n}$.

b) Define $\mathcal{C}_n(u) := \sqrt{n}(\varphi_n(u) - \varphi(u))$ and conclude from a) that $\mathbb{E}[\mathcal{C}_n(u)\overline{\mathcal{C}_n(v)}] = \varphi(u - v) - \varphi(u)\varphi(-v)$. Show that in combination with $\overline{\mathcal{C}_n(u)} = \mathcal{C}_n(-u)$ this completely determines the covariance structure of all finite-dimensional distributions, i.e., for all $u_1, \ldots, u_k$ the covariance of $\mathrm{Re}(\mathcal{C}_n(u_1)), \mathrm{Im}(\mathcal{C}_n(u_1)), \ldots, \mathrm{Re}(\mathcal{C}_n(u_k)), \mathrm{Im}(\mathcal{C}_n(u_k))$.

c) Define Γ to be the centred complex-valued Gaussian process with $\Gamma(-u) = \overline{\Gamma(u)}$ and $\mathrm{Cov}_{\mathbb{C}}(\Gamma(u), \Gamma(v)) = \varphi(u - v) - \varphi(u)\varphi(-v)$. Use the multivariate central limit theorem to show that $\mathcal{C}_n \xrightarrow{fidi} \Gamma$, i.e., all finite-dimensional distributions converge weakly.

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 8

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 8.1. Let $R > 8\sqrt{d}$ for $d \in \mathbb{N}$. Suppose that the empirical characteristic process $\mathcal{C}_n$ satisfies uniformly for $n \in \mathbb{N}$ and $K \geq 2$

$$\mathbb{P}\left(\max_{u \in [-K,K]^d} |\mathcal{C}_n(u)| \geq R\sqrt{\log(nK^2)} \right) \leq C(\sqrt{n}K)^{(64d-R^2)/(64d+64)}.$$

Show that the empirical characteristic function converges uniformly on compact sets in L^p, $p \geq 1$, to the true characteristic function with rate $(\log(n)/n)^{1/2}$.

Ex. 8.2. The following result is known as Hoeffding's inequality: If $X_1, \dots, X_n$ are mean zero independent random variables taking values in $[b_i, c_i]$ for constants $b_i < c_i$, $i = 1, ..., n$, respectively, then for $u > 0$

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i > u \right) \leq \exp\left(-\frac{2u^2}{\sum_{i=1}^{n}(c_i - b_i)^2} \right)$$

of which an obvious consequence is (why?)

$$\mathbb{P}\left(\left| \sum_{i=1}^{n} X_i \right| > u \right) \leq 2\exp\left(-\frac{2u^2}{\sum_{i=1}^{n}(c_i - b_i)^2} \right).$$

Provide a proof of this inequality. [Hint: You may find it useful to first prove the auxiliary result $\mathbb{E}[\exp(vX_i)] \leq \exp(v^2(c_i - b_i)^2/8)$ for $v > 0$, and then use Markov's inequality in conjunction with a bound for the moment generating function of $v\sum X_i$.]

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 9

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 9.1. Let K be a bounded function on $\mathbb{R}$ such that $(1/A)\int_0^A K(x)\,\mathrm{d}x \to 0$ as $A \to \pm\infty$.

a) For $f = c\mathbb{1}_{[\alpha,\beta]}$ with $c \in \mathbb{R}$ and $0 \leq \alpha < \beta < \infty$ show that $\int_0^\infty f(x)K(\lambda x)\,\mathrm{d}x \to 0$ as $\lambda \to \pm\infty$.

b) For $f \in L^1([0,\infty))$ prove that $\int_0^\infty f(x)K(\lambda x)\,\mathrm{d}x \to 0$ as $\lambda \to \pm\infty$. [Hint: You may use that for every $\epsilon > 0$ there exists a finite linear combination g_ϵ of functions as in a) such that $\int_0^\infty |f(x) - g_\epsilon(x)|\,\mathrm{d}x < \epsilon$.]

c) For $f \in L^1(\mathbb{R})$ conclude that $\int_{-\infty}^\infty f(x)K(\lambda x)\,\mathrm{d}x \to 0$ as $\lambda \to \pm\infty$.

d) Show the Riemann–Lebesgue lemma: If $f \in L^1(\mathbb{R})$, then $\int_\mathbb{R} f(x)e^{i\lambda x}\,\mathrm{d}x \to 0$ as $\lambda \to \pm\infty$.

Ex. 9.2. Let $\tilde{w}^{U_n}(u) := (1/U_n)\tilde{w}(u/U_n)$, where $\tilde{w}$ is a continuous function, supported on $[0,1]$ with $\tilde{w}(u) > 0$ for $u \in (0,1)$. Consider the optimisation problem

$$(\sigma_n^2, \lambda_n) := \mathrm{argmin}_{(\sigma^2,\lambda)} \int_0^\infty \tilde{w}^{U_n}(u)(\mathrm{Re}\,\psi_n(u) + \sigma^2 u^2/2 + \lambda)^2\,\mathrm{d}u.$$

a) Show that it is solved by $\sigma_n^2 = \int_0^\infty w_\sigma^{U_n}(u)\mathrm{Re}\ \psi_n(u)\,\mathrm{d}u$ and $\lambda_n = \int_0^\infty w_\lambda^{U_n}(u)\mathrm{Re}\ \psi_n(u)\,\mathrm{d}u$ and derive the form of $w_\sigma^{U_n}$ and $w_\lambda^{U_n}$ in terms of $\tilde{w}^{U_n}$. Verify that $w_\sigma^{U_n}(u) = U_n^{-3}w_\sigma^1(u/U_n)$ and $w_\lambda^{U_n}(u) = U_n^{-1}w_\lambda^1(u/U_n)$.

b) Derive the identities $\int_0^{U_n}(-u^2/2)w_\sigma^{U_n}(u)\,\mathrm{d}u = 1$, $\int_0^{U_n} w_\sigma^{U_n}(u)\,\mathrm{d}u = 0$, $\int_0^{U_n}(-1)w_\lambda^{U_n}(u)\,\mathrm{d}u = 1$ and $\int_0^{U_n}(-u^2/2)w_\lambda^{U_n}(u)\,\mathrm{d}u = 0$.

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 10

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 10.1. Denote by $\mathcal{F}f$ the Fourier transform of f. For an integer $s \geq 0$ suppose that v satisfies $\max_{k\in\{0,1,\dots,s\}} \|v^{(k)}\|_{L^2(\mathbb{R})} \leq C$ and $\|v^{(s)}\|_\infty \leq C$ for some $C > 0$, where $v^{(k)}$ is the k-th derivative of v. Let w_γ and w_γ be functions supported on $[0,1]$ such that $w_\gamma(\bullet)/(\bullet)^s$, $w_\lambda(\bullet)/(\bullet)^s \in L^2(\mathbb{R})$ and $\mathcal{F}[w_\gamma(\bullet)/(\bullet)^s]$, $\mathcal{F}[w_\lambda(\bullet)/(\bullet)^s] \in L^1(\mathbb{R})$. Define $w_\gamma^{U_n}(u) = U_n^{-2} w_\gamma(u/U_n)$ and $w_\lambda^{U_n}(u) = U_n^{-1} w_\lambda(u/U_n)$. Show that

$$\left|\int_0^\infty w_\gamma^{U_n}(u)\mathrm{Im}(\mathcal{F}\nu(u))\,\mathrm{d}u\right| \lesssim U_n^{-(s+2)},$$
$$\left|\int_0^\infty w_\lambda^{U_n}(u)\mathrm{Re}(\mathcal{F}\nu(u))\,\mathrm{d}u\right| \lesssim U_n^{-(s+1)}.$$

Ex. 10.2. Denote by $\mathcal{F}f$ the Fourier transform of f. For an integer $s \geq 1$ suppose that v satisfies $\max_{k\in\{0,1,\dots,s\}} \|v^{(k)}\|_{L^2(\mathbb{R})} \leq C$ and $\|v^{(s)}\|_\infty \leq C$ for some $C > 0$, where $v^{(k)}$ is the k-th derivative of v. Let w be a function supported on $[-1,1]$ such that $(1-w(\bullet))/(\bullet)^s \in L^2(\mathbb{R})$ and $\mathcal{F}[(1-w(\bullet))/(\bullet)^s] \in L^1(\mathbb{R})$. Show that

$$\left\|\mathcal{F}^{-1}[(1-w(\bullet/U_n))\mathcal{F}v(\bullet)]\right\|_\infty \lesssim U_n^{-s}.$$

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 11

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 11.1. Let $X_1, X_2, \ldots, X_n$ i.i.d. random variables. Let φ_n be the empirical characteristic function of $X_1, X_2, \ldots, X_n$ and $\varphi_n^{(k)}$ the k-th derivative of φ_n. For complex-valued random variables Z we define $\mathrm{Var}_{\mathbb{C}}(Z) = \mathbb{E}[|Z - \mathbb{E}[Z]|^2]$. Show that for $k = 0, 1, 2$ and for all $u \in \mathbb{R}$

$$\mathrm{Var}_{\mathbb{C}}\left(\varphi_n^{(k)}(u)\right) \leq \frac{1}{n}\,\mathbb{E}\left[X_1^{2k}\right]$$

Ex. 11.2. Let X be a one-dimensional Lévy process with Lévy measure ν satisfying $\int_{\mathbb{R}} x^2 \,\mathrm{d}\nu(x) < \infty$. Let $\mathbb{P}_\Delta$ be the probability measure of X_Δ. Show that for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$

$$\int_{\mathbb{R}} f(x)x^2 \frac{\mathrm{d}\,\mathbb{P}_\Delta(x)}{\Delta} \to \sigma^2 f(0) + \int_{\mathbb{R}} f(x)x^2 \,\mathrm{d}\nu(x) \quad \text{as } \Delta \to 0.$$

[You may use the following fact: For a sequence of probability measures $\mathbb{P}, \mathbb{P}_1, \mathbb{P}_2 \ldots$ on $\mathbb{R}$ with characteristic functions $\varphi, \varphi_1, \varphi_2, \ldots$ a consequence of Lévy's continuity theorem is that the pointwise convergence $\varphi_n \to \varphi$ as $n \to \infty$ implies that $\mathbb{P}_n$ converge weakly to $\mathbb{P}$ as $n \to \infty$.]

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 12

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 12.1. Let X and ϵ be independent real-valued random variables whose distributions are absolutely continuous with respect to the Lebesgue measure with Lebesgue densities p_X and p_ϵ, respectively. Let φ_ϵ and φ_Y be the characteristic functions of ϵ and $Y = X + \epsilon$, respectively. Let $\varphi_{Y,n}$ be the empirical characteristic function of n i.i.d. copies of Y. Suppose $\varphi_\epsilon \neq 0$ for all $u \in \mathbb{R}$ and let $M_U := \sup_{|u|\leq U} |1/\varphi_\epsilon|$. Assume for an integer $s \geq 1$ that $p_X^{(k)} \in L^2(\mathbb{R})$ for all $k \in \{0, 1, \ldots, s\}$. Define

$$\hat{p}_X = \mathcal{F}^{-1}\left[\frac{\varphi_{Y,n}}{\varphi_\epsilon} \mathbb{1}_{[-U_n,U_n]}\right].$$

Show that

$$\|\hat{p}_X - p_X\|_{L^2(\mathbb{R})} = O_{\mathbb{P}}\left(U_n^{-s} + M_{U_n} U_n^{1/2} n^{-1/2}\right).$$

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 13

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 13.1. Let $N = (N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda \geq 0$. For $\Delta > 0$ we define $Z_i := N_{i\Delta} - N_{(i-1)\Delta}$ with $i = 1, 2, \dots, n$.

a) Compute the maximum likelihood estimator of λ given the observations $\mathbb{1}_{Z_i \neq 0}$, $i = 1, \dots, n$. Compare to the nonlinear estimator $\tilde{\lambda}_n$ from the lecture.

b) Let

$$Y_i := \begin{cases} 0 & \text{if } Z_i = 0, \\ 1 & \text{if } Z_i = 1, \\ 2 & \text{if } Z_i \geq 2, \end{cases}$$

for $i = 1, \dots, n$.

Find the log-likelihood for estimating λ from the observations $Y_1, Y_2, \dots, Y_n$. Set its derivative equal to zero and approximate the solution of the resulting equation for high-frequency observations $\Delta \to 0$ as $n \to \infty$. Compare the resulting estimator to the linear estimator $\hat{\lambda}_n$ from the lecture.

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 14

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 14.1. Suppose that $(X_i)_{i=1}^n$ are zero-mean and independent random variables such that, for some fixed $q \geq 1$, they satisfy the moment bound $(\mathbb{E}[|X_i|^{2q}])^{\frac{1}{2q}} \leq K_q$. Show that

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n X_i\right| \geq \delta\right) \leq B_q \left(\frac{1}{\sqrt{n}\delta}\right)^{2q} \qquad \text{for all } \delta > 0,$$

where B_q is a universal constant depending only on K_q and q. [Hint: You may use Rosenthal's inequality.]

Ex. 14.2. Let X be a real-valued Lévy processes. Let $D := [a, b] \subseteq \mathbb{R} \setminus \{0\}$ and

$$\mathcal{S} := \{\beta_1\varphi_1 + \cdots + \beta_d\varphi_d \,|\, \beta_1, \ldots, \beta_d \in \mathbb{R}\},$$

where φ_j with $j = 1, \ldots, d$ are ν-a.e. continuous, bounded functions which have support in D and are orthonormal with respect to the inner product $\langle p, q\rangle := \int_D p(x)q(x)\,\mathrm{d}x$.
Let $0 = t_0 < t_1 < \cdots < t_n$ and define

$$\hat{\rho}(x) := \sum_{j=1}^d \hat{\beta}(\varphi_j)\varphi_j(x) \qquad \text{with} \quad \hat{\beta}(\varphi_j) := \frac{1}{t_n}\sum_{k=1}^n \varphi_j\left(X_{t_k} - X_{t_{k-1}}\right).$$

Show that $\hat{\rho}$ is the unique solution of the minimisation problem

$$\min_{f\in\mathcal{S}} \gamma_D(f),$$

where $\gamma_D : L^2(D, \mathrm{d}x) \to \mathbb{R}$ is given by

$$\gamma_D(f) := -\frac{2}{t_n}\sum_{k=1}^n f(X_{t_k} - X_{t_{k-1}}) + \int_D f^2(x)\,\mathrm{d}x.$$

STATISTICS FOR STOCHASTIC PROCESSES - EXERCISE SHEET 15

Mastermath, Spring Semester 2020, Jakob Söhl (j.soehl@tudelft.nl)

Ex. 15.1. Let X be a one-dimensional Lévy process with Lévy measure ν. Let φ have support in $[c,d] \subseteq \mathbb{R}_{>0}$ and let $\varphi|_{[c,d]}$ be continuous with continuous derivative.

a) Show that

$$\mathbb{E}[\varphi(X_\Delta)] = \varphi(c)\,\mathbb{P}(X_\Delta \geq c) + \int_c^\infty \varphi'(u)\,\mathbb{P}(X_\Delta \geq u)\,\mathrm{d}u,$$
$$\int_{-\infty}^\infty \varphi(x)\,\mathrm{d}\nu(x) = \varphi(c)\nu([c,\infty)) + \int_c^\infty \varphi'(u)\nu([u,\infty))\,\mathrm{d}u.$$

b) Derive further

$$\left|\frac{\mathbb{E}[\varphi(X_\Delta)]}{\Delta} - \int_{-\infty}^\infty \varphi(x)\,\mathrm{d}\nu(x)\right| \leq \left(|\varphi(c)| + \int_c^d |\varphi'(u)|\,\mathrm{d}u\right) M_\Delta([c,d]),$$

where $M_\Delta([c,d]) := \sup_{y\in[c,d]} |\frac{1}{\Delta}\,\mathbb{P}(X_\Delta \geq y) - \nu([y,\infty))|$.