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**Column** Piet takes his chance

# Buffon's needles and noodles

**Piet Groeneboom regularly writes a column on statistical topics in this magazine.**

**The dual problem and Oberwolfach**

What mathematical subject could be of interest to secondary school pupils? My colleague Chris Klaassen and I had (completely independently) the same idea: Buffon's needle problem. Chris talked about it in a lesson for pupils of the Leiden city gymnasium during his study and I wrote about it in a flyer for secondary schools of the Mathematical Institute of the University of Amsterdam in 1987. The flyer was intended to seduce the pupils of these schools to study mathematics (at the University of Amsterdam).

I recently understood from Chris that his main conclusion after the lesson at the Leiden school was that any desire to be a teacher at a secondary school had left him. And my own conclusion after writing about Buffon in the flyer of the Mathematics Institute of the University of Amsterdam was that I should perhaps write about something else if I would have to do it again.

After I had written the draft of my contribution on Buffon's needle for the flyer I received visits from nearly all my colleagues at the mathematics institute who told me that they thought it was too difficult for secondary school children. Only my colleague Ietje Paalman said that she thought it was an enjoyable piece to read, but also thought it was too difficult. When I moved from Amsterdam to Delft University in 1988 my contribution to the flyer was immediately replaced by an interview with Annoesjka Cabo (who wrote a dissertation on stochastic geometry, though, of which the Buffon needle problem is a kind of beginning).

So let's try Buffon's needle problem again, this time aimed at readers who may be somewhat beyond high school age. And conveniently in this magazine where Casper Albers, my predecessor as columnist, has already mentioned this problem and laid out the standard solution using trigonometry. But for this article I'm going to look at some altogether more interesting pathways to the proof.

My preferred proof (the proof I also tried to explain in the flyer for the secondary schools in Holland) is sketched in the introduction of the remarkable book on combinatorial integral geometry [2] by the Armenian mathematician Rouben Ambartzumian. Rouben is a recipient of the Rollo Davidson Prize (1982). I met him in Oberwolfach. He had a lot of jokes about mathematicians in Moscow, which resembled the Dutch jokes about Belgians (and possibly the Belgian jokes about the Dutch).

Now that I am mentioning Oberwolfach again, I realize that some readers might not know what I am talking about. The 'Mathematisches Forschungsinstitut Oberwolfach' (MFO) originally consisted (until 1969) of one building, 'the old castle', on a hill outside the village Oberwolfach in the Schwarzwald, Germany, where mathematicians could be invited to stay for a week to discuss a particular subject in mathematics. Then a new guest house and bungalows were constructed (in total, 3 buildings). In 1975 the old castle was replaced by a modern conference and library building.

It was founded during the Second World War (1944!). Nowadays there is one building where the bedrooms are and meals are served, and the library building is for the lectures. In the latter building there is also a billiard table, a ping pong room and a music room. The music room has a grand piano and several string instruments (not enough to play string quartet, though, but people often bring their own instrument) and also a pretty good choice of music scores. There is also a choice of wines and other drinks in this building.

Originally, one could only stay there by being the organizer of a workshop or by being invited by the organizers. Presently there are more buildings for guests and also the rules for staying there have somewhat changed. For example, one can try to get a grant for staying there with a colleague to work on a special project ('Oberwolfach Research fellows', previously called 'Research In Pairs', RIP). The door to one's bedroom had no key, "mathematicians trust one another" (this has changed too, since 2008 it is possible to lock the door from the inside, apartments for longer stays also have different rules).

Depending on the type of mathematicians gathering for such a week in Oberwolfach, there might be a lot of wine drinking and going to bed in the middle of the night/early morning or early to bed, early to rise. The week on stochastic geometry where I met Rouben Ambartzumian was a week of the first type. I had guessed this correctly beforehand (weeks of this type are very exhausting) and did not stay for the next week, for which I was also invited (this was a week on Lie algebras, if I remember correctly, where Tom Koornwinder was one of the organizers). So I do not know what the habits of this particular group were.

Anyway, coming back to Ambartzumian's sketch of proof of the probability in Buffon's needle problem, I now quote the first two paragraphs of the introduction of his book.

> "The idea of introducing measures into the space of lines in the plane was already implicit in Buffon's classical *needle problem*. Let us recall its formulation. The plane is ruled by a fixed lattice of parallel lines unit distance apart. A needle $\nu$ of length $|\nu|<1$ is 'thrown' at random onto the plane. What is the probability of the event that in its final position the needle will be intersected by a line of the lattice?
>
> In an equivalent formulation, needle and lattice exchange roles and one assumes that it is now the needle which is fixed in the plane with the lattice being thrown down at random."

When I first read this, "picking up the lines, throwing them on the non-moving needle", I liked this idea very much. And I thought this might also appeal to the secondary school pupils. Whether it did I do not know. In any case it did not appeal to my colleagues at the Mathematics Institute in Amsterdam.

The first bottleneck is the introduction of a measure on the space of lines. The invariant measure for lines in the plane is what Lebesgue measure ('area') is for points in the plane. The measure is invariant under translation and rotation. I tried to explain this concept in the flyer and this was something that my colleagues found particularly offensive. I said something like "invariant measure for lines is what area is for points". Unfortunately, I do not have the flyer anymore, so I do not know what I said exactly.

One would think that this flyer, which presumably was sent to all Dutch schools, should be traceable. I indeed contacted the Mathematics Institute of the University of Amsterdam in view of the present column, but the director of the institute told me that they could not look thoroughly for it because they were understaffed. This is the more regrettable, since Tobias Baanders, then at the Mathematical Centre (CWI), drew a very good picture for my contribution, where one could see a soldier (or officer?) of the American civil war with one arm in a sling, determining $\pi$ with his non-wounded arm by throwing a stick on a rotating wooden disc (see below for the occurrence of $\pi$ in this context). This described a historical event (which I had picked up from a book on 250 years of Buffon's needle).

Returning to Ambartzumian's book: he continues his exposition in the Introduction by saying:

> "Without loss of generality one may assume that the needle lies within some fixed open disc $K$ of unit diameter. Then in all possible outcomes of the lattice-throwing experiment the disc $K$ is intersected by exactly one line of the lattice, if we assume that the case of tangency is 'impossible' (i.e., of probability zero). Since other lines of the lattice now play no role, we may fix attention to this single line, $g_K$ say, intersecting $K$, and Buffon's original problem is now replaced by the following one: what is the probability $p$ that the random line $g_K$ intersecting $K$ should also intersect the needle? We may refer to this as the *dual* problem to the classical Buffon's needle problem."

And then, skipping a paragraph which is not directly relevant:

> "In the classical solution of Buffon's problem it is assumed that the centre of the needle and its orientation are independently and uniformly distributed, that is that the projection of the centre onto a line perpendicular to the lines of the lattice is uniformly distributed on some segment of unit length, and the angle between the line containing the needle and the lines of the lattice is independently and uniformly distributed on $(0,\pi)$. With these assumptions
>
> $$p=\frac{2}{\pi}|\nu|.$$
>
> (This example is the earliest instance of the calculation of a 'geometrical probability'.)
>
> To this result corresponds the following solution of the dual problem: there is a unique distribution $P$ of the random lines $g_K$ such that
>
> $$P\{g_K \text{ intersects } \nu\}=\frac{2}{\pi}|\nu|, \tag{1}$$
>
> for every needle $\nu$ within $K$. This distribution $P$ is proportional to the restriction, to the set of lines intersecting $K$, of the so-called *invariant measure* on the lines of the plane."

Ambartzumian does not explain how the $(2/\pi)|\nu|$ arises in the dual problem, but I will explain this now. There is the following 'easy to believe' theorem in stochastic geometry.

**Theorem 1.** *Let $A\subset B$ be two bounded convex sets in the plane. Then the probability that a random line meets $A$, given that it meets $B$ is given by*

$$p=\frac{\ell(\partial A)}{\ell(\partial B)}, \tag{2}$$

*where $\partial A$ $(\partial B)$ is the boundary of $A$ $(B)$ and $\ell$ denotes length.*

The length of the boundary of the needle $\nu$, as a degenerate convex set in the plane, is $2|\nu|$ in this interpretation. In measuring the length of the boundary we consider a lower side and an upper side. If we apply this to the computation of the probability on the left in (1) we get: $2|\nu|$ (length of boundary of needle) divided by $\pi$ (length of circumference of circle with diameter $d=1$).

This argument also immediately gives that if the distance between the lines of the lattices is $d$ instead of $1$, the probability is given by

$$\frac{2|\nu|}{\pi d},$$

if $|\nu|<d$.

### Buffon's noodles

In [1] the 'Buffon's noodle' proof is given. This proof starts with the trivial observation that the probability we are looking for can also be interpreted as an expectation of number of crossings. This phenomenon is due to the 'hit or miss' character of the event of hitting a line of the lattice, the expectation of 'the number of crossings' is $p \cdot 1 + 0 \cdot (1-p) = p$, if $p$ is the probability of crossing.

We can then consider the expectation of the number of crossings of lines of the lattice by a 'polygonal needle' $P$ (a *noodle*), consisting of several needles. By the linearity of the expectation operator, the expected number of crossings of lines in the lattice by this polygonal needle will be proportional to its length. In expectation notation we get, for some constant $c > 0$,

$$\mathbb{E}\{\#\text{crossings of } P \text{ with lines of lattice}\} = c\ell(P) \qquad (3)$$

if $\ell(P)$ is the length of the polygonal needle $P$.

In particular, we can consider polygonal needles approaching a circle with diameter $d$ if the distance between the lines of the lattice is $d$. A circle with diameter $d$, thrown down on this lattice, will always have *two* crossings. Considering polygonal needles $P_n$ of this type, approaching the circle, we get from (3), disregarding circles which hit two lines and have tangents there (this has probability zero)

$$\lim_{n \to \infty} c\ell(P_n) = c\pi d = 2.$$

implying $c = 2/(\pi d)$. This in turn implies, again by (3), that for our simple (degenerate polygonal) needle $\nu$ with length $|\nu|$ the expected number of crossings is $2|\nu|/(\pi d)$. And hence, by the trivial observation at the beginning of this argument, this is also the probability sought for.

This is Barbier's solution [4], which also avoids any calculation with sines or cosines. Still I prefer Ambartzumian's proof. It would be great if someone could still locate the flyer of the Mathematics Institute of the University of Amsterdam from (around) 1987 with Ambartzumian's proof and the drawing of Tobias Baanders.

### Postscriptum

After having written this column and having sent it to Tobias Baanders, he kindly made a new drawing, based on a silly sketch I made in the train from my memory of his drawing in the flyer. So here is a Tobias Baanders drawing after all with this column. He depicts the situation where the needle is longer than the distance between the lines, and one counts the number of crossings, which in principle leads to more correct digits in the determination of $\pi$ in this way (which is a rather pointless exercise anyway).

Determining $\pi$

Illustration: Tobias Baanders

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### References

1 Martin Aigner and Günter M. Ziegler, *Proofs from The Book*, Springer, 2018, sixth edition.

2 R.V. Ambartzumian, *Combinatorial Integral Geometry: With Applications to Mathematical Stereology*, Wiley Series in Probability and Mathematical Statistics: Tracts on Probability and Statistics, Wiley, 1982.

3 E. Barbier, Note sur le problème de l' aiguille et le jeu du joint couvert, *J. Mathematiques Pures et Appliquées* 5(4) (1860), 273–286.